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APPLICATION OF GAUGE THEORY OF DEFECTS TO FRACTURE

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Abstract—In this paper, we give E. Noether's symmetrical theorem in Kadic and Edelen's gauge theory of defect continuum and from this theorem obtain easily the field equations of defect continuum. The conservation law under the space translation group is applied to the study of the fracture of materials.

INTRODUCTION

The subsequent development of defect theory can be divided roughly into three periods. The third period covers the years after 1978. In this period, the theory was developed for gauge theory of defects which contains two parts, namely the Lagrangian and geometrical theory.

In the Lagrangian gauge theory of defects, the earliest paper was published by Fa (1978). Later, there were papers published independently by Fa (1981), Golebiewska-Lasota (1979) and Kadic and Edelen (1983). Among them, there were some works on transformations of the field equations and characteristic quantities of dislocation which made them invariant. In the gauge theory of defect continuum, the symmetrical theorem and the conservation laws have not been discussed. In particular, their application to studying fracture of materials is not found.

In this paper, we give Noether's symmetrical theorem and the conservation laws on a gauge theory of defect continuum. The conservation law under the space translation group is applied to studying fracture of materials. A new conservation integral may be considered as an extension of Eshelby's energy–momentum tensor to a more general case.

PREPARED KNOWLEDGE (KADIC AND EDELEN, 1983)

Let y^a ($a = 1, 2, 3, 4$) be the global coordinates of the reference configuration and $y^4 = t$ be time, the generalized spatial coordinates of the local coordinate system are

$$\hat{x} = (x^1, x^2, x^3, x^4)^T = (x^1, x^2, x^3, 1)^T.$$

The generalized Cauchy strain tensors are

$$\hat{C}_{ab} = \partial_a \hat{x}^T \cdot \partial_b \hat{x} = \partial_a \mathbf{x}^T \cdot \partial_b \mathbf{x} = C_{ab}. \tag{1}$$

Therefore, $L(\hat{C}_{ab}) = L(C_{ab})$, where L is the Lagrangian of a classical elastic field. The Lagrangian function $L(\hat{C}_{ab})$ is invariant under the global gauge group $G_0 = SO(3)_0 \rhd T(3)_0$.

Let us now consider the local gauge invariance. According to the gauge theory (Abers and Lee, 1973), we introduce the gauge potential W and ϕ such as

$$B_a^\alpha = \partial_a x^\alpha + \gamma_{A\beta}^\alpha x^\beta W_a^A + \phi_a^\alpha \tag{2}$$

where $A = 1, 2, 3$.

The Lagrangian function of defect continuum can be written as

$$L = L_0 - S_1 L_1 - S_2 L_2 \tag{3}$$

where

$$L_1 = \tfrac{1}{2}\delta_{\alpha\beta} D^{\alpha}_{ab} K^{ac} K^{bd} D^{\beta}_{cd} \tag{4}$$

and

$$L_2 = \tfrac{1}{2} C_{\alpha\beta} F^{\alpha}_{ab} g^{ac} g^{bd} F^{\beta}_{cd} \tag{5}$$

in which

$$g^{AB} = -\delta^{AB}, \quad g^{44} = 1/\zeta, \quad g^{ab} = 0, \quad \text{for } a \neq b$$

and

$$K^{AB} = -\delta^{AB}, \quad K^{44} = 1/\xi, \quad K^{ab} = 0, \quad \text{for } a \neq b.$$

S_1 and S_2 are coupled constants. $C_{\alpha\beta}$ are the components of the Cartan–Killing metric of the subgroup $SO(3)$. L_1 and L_2 are the Lagrangian functions of the ϕ- and W-fields, respectively.

For the disclination field, we have

$$F^{\alpha}_{ab} = \partial_a W^{\alpha}_b - \partial_b W^{\alpha}_a + C^{\alpha}_{\beta\gamma} W^{\beta}_a W^{\gamma}_b. \tag{6}$$

For the dislocation field, we have

$$D^{\alpha}_{ab} = \partial_a \phi^{\alpha}_b - \partial_b \phi^{\alpha}_a + \gamma^{\alpha}_{\sigma\beta}[W^{\sigma}_a \phi^{\beta}_b - W^{\sigma}_b \phi^{\beta}_a + F^{\sigma}_{ab} x^{\beta}]. \tag{7}$$

Therefore, the total Lagrangian function of defect continuum is

$$L = L(y^a, x^{\alpha}, x^{\alpha}_{,a}, W^{\alpha}_a, W^{\alpha}_{a,b}, \phi^{\alpha}_a, \phi^{\alpha}_{a,b}) \tag{8}$$

where y^a are reference coordinates, and x^{α}, $x^{\alpha}_{,a}$ are field quantities. The total Lagrangian function is invariant under the group $G = SO(3) \rhd T(3)$. Therefore, the action functional follows as

$$S = \int_V L(y^a, x^{\alpha}, x^{\alpha}_{,a}, W^{\alpha}_a, W^{\alpha}_{a,b}, \phi^{\alpha}_a, \phi^{\alpha}_{a,b})\, \mathrm{d}V \tag{9}$$

which is invariant under the local group $G = SO(3) \rhd T(3)$.

THE SYMMETRICAL THEOREM

In this section, we give Noether's symmetrical theorem of defect continuum.

We consider a continuous transformation group with one single parameter

$$\hat{G}\begin{cases} y'^a = y^a + f^a\eta + 0^a_1(\eta^2) \\ x'^{\alpha} = x^{\alpha} + g^{\alpha}\eta + 0^{\alpha}_2(\eta^2) \\ W'^{\alpha}_a = W^{\alpha}_a + P^{\alpha}_a\eta + 0^{\alpha}_3(\eta^2) \\ \phi'^{\alpha}_a = \phi^{\alpha}_a + q^{\alpha}_a\eta + 0^{\alpha}_4(\eta^2) \end{cases} \tag{10}$$

where η is a parameter, $\alpha = 1, 2, 3$; $a = 1, 2, 3, 4$ and the quantities

$$f^a = \left.\frac{\partial y'^a}{\partial \eta}\right|_{\eta=0}, \quad g^\alpha = \left.\frac{\partial x'^\alpha}{\partial \eta}\right|_{\eta=0}, \quad p_a^\alpha = \left.\frac{\partial W_a'^\alpha}{\partial \eta}\right|_{\eta=0}, \quad q_a^\alpha = \left.\frac{\partial \phi_a'^\alpha}{\partial \eta}\right|_{\eta=0}. \tag{11}$$

It is obvious that we can obtain some kind of transformation group by assigning f^a, g^α, p_a^α and q_a^α some special values.

We give Noether's symmetrical theorem as follows (Konopleva and Popov, 1981).

If the action functional S has infinitesimal invariance under the local gauge group $\hat{G}$, there exists

$$\frac{\delta L}{\delta x^\alpha}\bar{\delta}x^\alpha + \frac{\delta L}{\delta W_a^\alpha}\bar{\delta}W_a^\alpha + \frac{\delta L}{\delta \phi_a^\alpha}\bar{\delta}\phi_a^\alpha = -J^a_{,a} \tag{12}$$

where

$$J^a = \frac{\partial L}{\partial x^\alpha_{,a}}\bar{\delta}x^\alpha + \frac{\partial L}{\partial W^\alpha_{b,a}}\bar{\delta}W_b^\alpha + \frac{\partial L}{\partial \phi^\alpha_{b,a}}\bar{\delta}\phi_b^\alpha \tag{13}$$

$$\begin{cases} \delta L/\delta x^\alpha = \partial L/\partial x^\alpha - (\partial L/\partial x^\alpha_{,a})_{,a} \\ \delta L/\delta W_a^\alpha = \partial L/\partial W_a^\alpha - (\partial L/\partial W^\alpha_{a,b})_{,b} \\ \delta L/\delta \phi_a^\alpha = \partial L/\partial \phi_a^\alpha - (\partial L/\partial \phi^\alpha_{a,b})_{,b} \end{cases} \tag{14}$$

and

$$\begin{cases} \bar{\delta}x^\alpha = \Delta x^\alpha - x^\alpha_{,a}\Delta y^a \\ \bar{\delta}W_a^\alpha = \Delta W_a^\alpha - W^\alpha_{c,a}\Delta y^c \\ \bar{\delta}\phi_a^\alpha = \Delta\phi_a^\alpha - \phi^\alpha_{c,a}\Delta y^c. \end{cases} \tag{15}$$

Proof. If the action functional S has infinitesimal invariance under the local gauge group $\hat{G}$, we have

$$\int_{V'} L(y^a + \Delta y^a, x^\alpha + \Delta x^\alpha, x^\alpha_{,a} + \Delta x^\alpha_{,a}, W_a^\alpha + \Delta W_a^\alpha, W^\alpha_{a,b} + \Delta W^\alpha_{a,b}, \phi_a^\alpha + \Delta\phi_a^\alpha, \phi^\alpha_{a,b} + \Delta\phi^\alpha_{a,b})\,\mathrm{d}V'$$
$$= \int_V L(y^a, x^\alpha, x^\alpha_{,a}, W_a^\alpha, W^\alpha_{a,b}, \phi_a^\alpha, \phi^\alpha_{a,b})\,\mathrm{d}V.$$

Since the transformation of the volume element is

$$\mathrm{d}V' = [1 + (\Delta y^a)_{,a}]\,\mathrm{d}V$$

and

$$L(y^a + \Delta y^a, x^\alpha + \Delta x^\alpha, x^\alpha_{,a} + \Delta x^\alpha_{,a}, W_a^\alpha + \Delta W_a^\alpha, W^\alpha_{a,b} + \Delta W^\alpha_{a,b}, \phi_a^\alpha + \Delta\phi_a^\alpha, \phi^\alpha_{a,b} + \Delta\phi^\alpha_{a,b})$$
$$= L + \frac{\partial L}{\partial y^a}\Delta y^a + \frac{\partial L}{\partial x^\alpha}\Delta x^\alpha + \frac{\partial L}{\partial x^\alpha_{,a}}\Delta x^\alpha_{,a} + \frac{\partial L}{\partial W_a^\alpha}\Delta W_a^\alpha + \frac{\partial L}{\partial W^\alpha_{a,b}}\Delta W^\alpha_{a,b} + \frac{\partial L}{\partial \phi_a^\alpha}\Delta\phi_a^\alpha + \frac{\partial L}{\partial \phi^\alpha_{a,b}}\Delta\phi^\alpha_{a,b}.$$

According to the infinitesimal invariance of the action functional we obtain

$$\frac{\partial L}{\partial y^d}\Delta y^d+\frac{\partial L}{\partial x^\alpha}\Delta x^\alpha+\frac{\partial L}{\partial W^\alpha_a}\Delta W^\alpha_a+\frac{\partial L}{\partial W^\alpha_{a,b}}\Delta W^\alpha_{a,b}+\frac{\partial L}{\partial \phi^\alpha_a}\Delta\phi^\alpha_a+\frac{\partial L}{\partial \phi^\alpha_{a,b}}\Delta\phi^\alpha_{a,b}+L(\Delta y^d)_{,d}=0$$

or

$$\frac{\partial L}{\partial x^\alpha}\Delta x^\alpha+\frac{\partial L}{\partial x^\alpha_{,a}}\Delta x^\alpha_{,a}+\frac{\partial L}{\partial W^\alpha_a}\Delta W^\alpha_a+\frac{\partial L}{\partial W^\alpha_{a,b}}\Delta W^\alpha_{a,b}+\frac{\partial L}{\partial \phi^\alpha_a}\Delta\phi^\alpha_a+\frac{\partial L}{\partial \phi^\alpha_{a,b}}\Delta\phi^\alpha_{a,b}$$
$$+L(\Delta y^d)_{,d}-\left(\frac{\partial L}{\partial x^\alpha}x^\alpha_{,c}+\frac{\partial L}{\partial x^\alpha_{,a}}x^\alpha_{,ac}+\frac{\partial L}{\partial W^\alpha_a}W^\alpha_{a,c}+\frac{\partial L}{\partial W^\alpha_{a,b}}W^\alpha_{a,bc}\right.$$
$$\left.+\frac{\partial L}{\partial \phi^\alpha_a}\phi^\alpha_{a,c}+\frac{\partial L}{\partial \phi^\alpha_{a,b}}\phi^\alpha_{a,bc}\right)\Delta y^c=0.$$

Finally, we have

$$\frac{\delta L}{\partial x^\alpha}\bar{\delta}x^\alpha+\frac{\delta L}{\delta W^\alpha_a}\bar{\delta}W^\alpha_a+\frac{\delta L}{\delta \phi^\alpha_a}\bar{\delta}\phi^\alpha_a+\left(\frac{\partial L}{\partial x^\alpha_{,a}}\bar{\delta}x^\alpha+\frac{\partial L}{\partial W^\alpha_{b,a}}\bar{\delta}W^\alpha_b+\frac{\partial L}{\partial \phi^\alpha_{b,a}}\bar{\delta}\phi^\alpha_b\right)_{,a}=0.$$

Noether's symmetrical theorem plays an important role in modern field theory. From it we can obtain the field equations of defect continuum, the conservation laws and the dynamical criterion of a singularity (dislocations or crack, etc.) motion.

Obviously, from Noether's symmetrical theorem, the field equations describing defect continuum are given by

$$\begin{cases}\dfrac{\partial L}{\partial x^\alpha}-\left(\dfrac{\partial L}{\partial x^\alpha_{,a}}\right)_{,a}=0\\[2ex] \dfrac{\partial L}{\partial W^\alpha_a}-\left(\dfrac{\partial L}{\partial W^\alpha_{a,b}}\right)_{,b}=0\\[2ex] \dfrac{\partial L}{\partial \phi^\alpha_a}-\left(\dfrac{\partial L}{\partial \phi^\alpha_{a,b}}\right)_{,b}=0.\end{cases}\tag{16}$$

In fact, the field equations, eqns (16), are the result of the symmetrical theorem. This method is different from Kadic and Edelen's theory.

APPLICATION TO FRACTURE

It is well known that Eshelby (1956) first gave a formula for the force acting on a general elastic singularity (Liebowitze, 1968) which is called the energy–momentum tensor. The J-integral of Rice (1968) is equal to the 2-D form of it. Now, we may extend it to a generalized case, a defect continuum, gauge theory of defects.

In order to give the application to fracture, we discuss the conservation law under the space translation group. We define

$$y'^d=y^d+C^d\varepsilon\tag{17}$$

as a spatial translation group, in which C is a constant vector and ε an infinitesimal quantity. Therefore

$$f^4=0,\quad f^i=C^i,\quad g^\alpha=p^\alpha_a=q^\alpha_a=0.\tag{18}$$

Substituting the above into Noether's symmetrical theorem, eqn (12), when x, W and ϕ are extremals, the conservation law under this group, can be given as

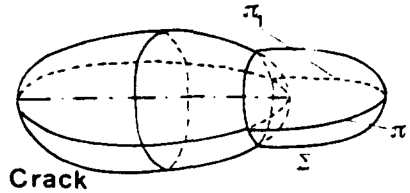

Fig. 1. The closed surface surrounding the crack tip.

$$\left(L\delta_{ak} - \frac{\partial L}{\partial x^\alpha_{,a}} x^\alpha_{,k} - \frac{\partial L}{\partial W^\alpha_{b,a}} W^\alpha_{b,k} - \frac{\partial L}{\partial \phi^\alpha_{b,a}} \phi^\alpha_{b,k}\right)_{,a} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x^\alpha} x^\alpha_{,k} + \frac{\partial L}{\partial W^\alpha_b} W^\alpha_{b,k} + \frac{\partial L}{\partial \phi^\alpha_b} \phi^\alpha_{b,k}\right). \tag{19}$$

Integrating eqn (19), we obtain

$$\int_\Sigma \left(L\delta_{ak} - \frac{\partial L}{\partial x^\alpha_{,a}} x^\alpha_{,k} - \frac{\partial L}{\partial W^\alpha_{b,a}} W^\alpha_{b,k} - \frac{\partial L}{\partial \phi^\alpha_{b,a}} \phi^\alpha_{b,k}\right) n_a \, \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \left(\frac{\partial L}{\partial x^\alpha} x^\alpha_{,k} + \frac{\partial L}{\partial W^\alpha_b} W^\alpha_{b,k} + \frac{\partial L}{\partial \phi^\alpha_b} \phi^\alpha_{b,k}\right) \mathrm{d}V.$$

The right-hand side of the equation is the energy release rate or the force acting on the singularity (dislocation, crack, etc.). Here we mainly aim at the crack propagation force, it can be expressed by the left-hand side of the equation. Take a closed surface surrounding the tip in Fig. 1. It consists of the free surface of the crack tip π_a, and a smooth surface π. The total driving force acting on the crack tip is

$$\int_\Sigma \left(L\delta_{ak} - \frac{\partial L}{\partial x^\alpha_{,a}} x^\alpha_{,k} - \frac{\partial L}{\partial W^\alpha_{b,a}} W^\alpha_{b,k} - \frac{\partial L}{\partial \phi^\alpha_{b,a}} \phi^\alpha_{b,k}\right) n_a \, \mathrm{d}s. \tag{20}$$

Because the surface of the crack is free, the total driving force acting on the crack can be expressed by an integral as

$$F_k = \int_\pi \left(L\delta_{ak} - \frac{\partial L}{\partial x^\alpha_{,a}} x^\alpha_{,k} - \frac{\partial L}{\partial W^\alpha_{b,a}} W^\alpha_{b,k} - \frac{\partial L}{\partial \phi^\alpha_{b,a}} \phi^\alpha_{b,k}\right) n_a \, \mathrm{d}s. \tag{21}$$

Therefore, we obtain a dynamical criterion of crack propagation in the defect continuum. Let the critical driving or energy release rate of a crack in the existence of dislocations and disclinations be F_{kc}. When

$$F_k > F_{kc}$$

the crack starts to propagate. F_{kc} is a material constant. F_k can also be written as

$$F_k = \int_\pi \left(L\delta_{ak} - \frac{\partial L_0}{\partial B^\alpha_a} x^\alpha_{,k} - \frac{S_1}{2}\delta_{\alpha\beta} K^{ac} K^{bd} D^\beta_{cd}(\phi^\alpha_{b,k} + x^{\beta}{}_{,\gamma}\gamma^{\alpha}_{\sigma\beta} W^\alpha_{b,k}) - \frac{S_2}{2} C_{\alpha\beta} g^{ac} g^{bd} F^\beta_{cd} W^\alpha_{b,k}\right) n_a \, \mathrm{d}s. \tag{22}$$

Because no limitation is given to the particular form of $L_0(B^\alpha_a)$, the above conclusion has a generalized significance.

In the case of defect free materials F_k is simplified as

$$F_k = \int_{\Sigma} \left(L_0\delta_{\alpha k} - \frac{\partial L_0}{\partial x^{\alpha}_{,\alpha}} x^{\alpha}_{,k} \right) n_\alpha \, ds. \tag{23}$$

It is easy to prove that F_k is a conservational integral. It can be rewritten as

$$F_k = \int_{\Sigma} (L_0\delta_{\alpha k} - \sigma_{\alpha\alpha} u_{\alpha,k}) n_\alpha \, ds. \tag{24}$$

Compared with Eshelby's energy-momentum tensor (Konopleva and Popov, 1981; Rice, 1968) for the motion of elastic singularity

$$T_{\alpha k} = L\delta_{\alpha k} - \sigma_{\alpha\alpha} u_{\alpha,k}. \tag{25}$$

F_k is an integral over the curved surface of a crack which expresses the force acting on the surface Σ in the direction of y^k.

Let $y^1 = x$, $y^2 = y$, $y^3 = z$, projecting F_k onto the x–y plane, we have

$$F_1 = \int_{\Gamma} (L \, dy - T_\alpha u_{\alpha,k} \, ds).$$

$T_\alpha = \sigma_{\alpha\alpha} n_\alpha$ is the driving force on Γ. It is the same as Rice's J-integral, the extension force acting on a crack tip. Therefore, the F-integral given in Fa (1981) and Golebiewska – Lasota and Edelen (1979) is the generalization of the elastic J-integral in the existence of dislocations and disclinations, and the criterion of fracture given in this paper is the generalization of the criterion of fracture on defect free materials.

DISCUSSION

Physical and mechanical researchers have been interested in treating the plastic region near a crack tip and some problems of dynamics. From the viewpoint of the physical mechanism of plastic deformation, the macro-plastic region was taken as the continuous distribution of microscopic defects. By virtue of the field theory, we present the F-integral, eqn (22).

In fact, the gauge field theory of defect continuum is a field theory of generalized continuum and Noether's theorem, which various conservation laws are derived from, plays an important role in modern field theory. For the static, the integral conservation laws are some path-independent integrals. However, for the dynamics, it is the same as the static, e.g. F-integral (22) denotes the energy release rate or the total driving force acting on the free surface of the crack, therefore defect fields correspond to providing a resistance to crack propagation.

In this paper, J-integral (10) is a special case of our F-integral based on the gauge theory of defect continuum which is different from BCS and Yokobory's theory. In particular, the macro-behaviours of mechanics and the micro-mechanism of physics are connected by the gauge theory of defects.

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